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On the connection coefficients and recurrence relations arising from expansions in series of Laguerre polynomials

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Abstract

A formula expressing the Laguerre coefficients of a general-order derivative of an infinitely differentiable function in terms of its original coefficients is proved, and a formula expressing explicitly the derivatives of Laguerre polynomials of any degree and for any order as a linear combination of suitable Laguerre polynomials is deduced. A formula for the Laguerre coefficients of the moments of one single Laguerre polynomial of certain degree is given. Formulae for the Laguerre coefficients of the moments of a general-order derivative of an infinitely differentiable function in terms of its Laguerre coefficients are also obtained. A simple approach in order to build and solve recursively for the connection coefficients between Jacobi–Laguerre and Hermite–Laguerre polynomials is described. An explicit formula for these coefficients between Jacobi and Laguerre polynomials is given, of which the ultra-spherical polynomials of the first and second kinds and Legendre polynomials are important special cases. An analytical formula for the connection coefficients between Hermite and Laguerre polynomials is also obtained.

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1. Introduction

The problem of approximating solutions of differential equations by spectral methods, known as Galerkin approximations, involves the projection onto the span of some appropriate set of basis functions, typically arising as the eigenfunctions of a singular Sturm–Liouville problem. The members of the basis may satisfy automatically the auxiliary conditions imposed on the problem, such as initial, boundary or more general conditions. Alternatively, these conditions may be imposed as constraints on the expansion coefficients, as in the Lanczos τ-method (Lanczos 1957).

It is well known (Canuto *et al* 1988) that the eigenfunctions of certain singular Sturm–Liouville problems allow the approximation of functions in $C^{\infty}[a, b]$ whose truncation error approaches zero faster than any finite negative power of the number of basis functions (retained modes) used in the approximation, as that number (order of truncation N) tends to ∞. This phenomenon is usually referred to as 'spectral accuracy' (Gottlieb and Orszag 1977).

It is of fundamental importance to know that the choice of the basis functions is responsible for the superior approximation properties of spectral methods when compared with the finite difference and finite element methods.

Spectral methods provide a computational approach which has achieved substantial popularity over the last three decades. They have gained new popularity in automatic computations for a wide class of physical problems in fluid and heat flow. The principal advantage of spectral methods lies in their ability to achieve accurate results with substantially fewer degrees of freedom.

Spectral methods have been used extensively in the solution of the boundary value problems and computational fluid dynamics, see for instance, Fox and Parker (1972), Gottlieb and Orszag (1977), Canuto *et al* (1988) and Doha and Abd-Elhameed (2002). In most of these applications, a formula is used that relates the expansion coefficients of derivatives appearing in the differential equation to those of the function itself. For the Galerkin and tau variants of the spectral methods, explicit expressions for the expansion coefficients for the solution are needed. Karageorghis (1998a) obtained an expression when the basis functions of expansion are shifted Chebyshev polynomials $T_n^*(x)$, $x \in [0, 1]$. A corresponding formula for Legendre polynomials $P_n(x)$, $x \in [-1, 1]$, is derived by Phillips (1988). Doha (1991) has obtained a more general formula when the basis functions are the ultraspherical polynomials $C_n^{(\alpha)}(x)$, $x \in [-1, 1]$, $\alpha \in \left(-\frac{1}{2}, \infty\right)$; formulae for the first and second kinds of Chebyshev polynomials and Legendre polynomials $T_n(x)$, $U_n(x)$ and $P_n(x)$ are given as special cases of $C_n^{(\alpha)}(x)$. A most general formula when the basis functions are the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $x \in [-1, 1]$, $\alpha > -1$, $\beta > -1$, is given in Doha (2002). Another formula when the basis functions are the Hermite polynomials is obtained in Doha (2003).

A more general situation which often arises in the numerical solution of differential equations with polynomial coefficients in spectral methods is the evaluation of the expansion coefficients of the moments of high-order derivatives of infinitely differentiable functions. A formula for the shifted Chebyshev coefficients of the moments of general-order derivatives of an infinitely differentiable function is given in Karageorghis (1998b). Corresponding results for Chebyshev polynomials of the first and second kinds, Legendre, ultraspherical and Hermite polynomials are given in Doha (1994), Doha and El-Soubhy (1995), Doha (1998) and Doha (2003) respectively.

Up to now, and to the best of our knowledge, many formulae corresponding to those mentioned previously are not known and traceless in the literature for the Laguerre expansions. This motivates our interest in such polynomials. Another motivation is that the theoretical and numerical analyses of numerous physical and mathematical problems very often require the expansion of an arbitrary polynomial or the expansion of an arbitrary function with its derivatives and moments into a set of orthogonal polynomials. This is in particular true (for Laguerre polynomials) in quantum mechanical studies of physical systems, where the equation of motion or Schrödinger equation is a second-order differential equation with polynomial coefficients. This is the case not only for the solution of the Schrödinger, Klein–Gordan and Dirac equations for the Coulomb field but also for many other potentials, as shown for example in Bagrov and Gitman (1990) and Nikiforov and Uvarov (1988).

The paper is organized as follows. In section 2, we give some properties of Laguerre polynomials. In section 3, we prove a theorem which relates the Laguerre expansion coefficients of the derivatives of a function in terms of its original expansion coefficients. An explicit expression for the derivatives of Laguerre polynomials of any degree and for any order as a linear combination of suitable Laguerre polynomials themselves is also deduced. A theorem which gives the Laguerre coefficients of the moments of one single Laguerre polynomial of any degree is considered in section 4. In section 5, we state and prove a theorem which expresses explicitly the Laguerre coefficients of the moments of a general-order derivative of an infinitely differentiable function in terms of its Laguerre coefficients. Application of these theorems for solving ordinary differential equations with varying coefficients, by reducing them to recurrence relations in the expansion coefficients of the solution, is given in section 6. A simple approach in order to build and solve recursively for the connection coefficients between two families of orthogonal polynomials as solutions of second-order differential equations is described in section 7.

2. Some properties of Laguerre polynomials

The Laguerre polynomials are a sequence of polynomials $\left\{L_n^{(\alpha)}(x), n = 0, 1, 2, \ldots\right\}$, each of degree n, satisfying the orthogonality relation

$$\int_0^\infty \mathrm{e}^{-x} x^\alpha L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\,\mathrm{d}x = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn} \qquad \alpha > -1. \tag{1}$$

It is worth mentioning that many properties of Laguerre polynomials may be found in Rainville (1960). The Laguerre polynomials may be generated by using Rodrigue's formula

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}\,\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\mathrm{e}^{-x}x^{n+\alpha}]. \tag{2}$$

The following two recurrence relations are of fundamental importance in developing the present work. These are

$$(n+1)L_{n+1}^{(\alpha)}(x) = (2n+\alpha+1-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) \qquad n = 0, 1, 2, \ldots \tag{3}$$

with $L_{-1}^{(\alpha)}(x) = 0$, and

$$L_n^{(\alpha)}(x) = D\left[L_n^{(\alpha)}(x) - L_{n+1}^{(\alpha)}(x)\right] \qquad n = 0, 1, 2, \ldots \tag{4}$$

where $D \equiv \frac{\mathrm{d}}{\mathrm{d}x}$. Note that the recurrence relation (3) may be used to generate the Laguerre polynomials starting from $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = (\alpha + 1 - x)$.

Suppose now we are given a function $f(x)$ which is infinitely differentiable in the interval $[0, \infty)$, then we can write

$$f(x) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \tag{5}$$

and for the qth derivative of $f(x)$,

$$f^{(q)}(x) = \sum_{n=0}^{\infty} a_n^{(q)} L_n^{(\alpha)}(x) \qquad a_n^{(0)} = a_n. \tag{6}$$

Moreover, if $f(x)$ satisfies

$$f(x) = O(\mathrm{e}^{\alpha x}) \qquad x \to \infty$$

for some $\alpha < \frac{1}{2}$, it can be shown (cf Gottlieb and Orszag (1977)) that the Laguerre expansion

$$f(x) = \sum_{n=0}^{N} a_n L_n^{(\alpha)}(x)$$

converges faster than algebraically as the number of terms $N \to \infty$.

3. Relations between the coefficients $a_n^{(q)}$ and a_n and the qth derivative of $L_n^{(\alpha)}(x)$

Theorem 1. *If $f(x)$ is infinitely differentiable and expanded as in (5), and the qth derivative of $f(x)$ is expressed as in (6), then*

$$a_n^{(q)} = \sum_{j=0}^{n} \binom{n-j+q-1}{q-1} a_j \tag{7}$$

and

$$D^q L_n^{(\alpha)}(x) = (-1)^q \sum_{j=0}^{n-q} \binom{n-j-1}{q-1} L_j^{(\alpha)}(x) \qquad n, q \geqslant 1. \tag{8}$$

Proof.

$$f^{(q+1)}(x) = \sum_{n=0}^{\infty} a_n^{(q+1)} L_n^{(\alpha)}(x)$$

and on differentiating (6), and making use of (4), we get

$$a_n^{(q+1)} - a_{n-1}^{(q+1)} = a_n^{(q)} \qquad n = 1, 2, \ldots, q \geqslant 0$$

which immediately gives

$$a_n^{(q+1)} = \sum_{j=0}^{n} a_j^{(q)}$$

and this in turn yields

$$a_n^{(1)} = \sum_{j=0}^{n} a_j \qquad a_n^{(2)} = \sum_{j=0}^{n} (n-j+1) a_j \qquad a_n^{(3)} = \sum_{j=0}^{n} \frac{(n-j+1)(n-j+2)}{2} a_j$$

and finally

$$a_n^{(q)} = \sum_{j=0}^{n} \frac{(n-j+1)(n-j+2)\cdots(n-j+q-1)}{(q-1)!} a_j$$

i.e.

$$a_n^{(q)} = \sum_{j=0}^{n} \binom{n-j+q-1}{q-1} a_j$$

which proves (7). From the properties of Laguerre polynomials, it can be easily shown that

$$DL_n^{(\alpha)}(x) = -\sum_{j=0}^{n-1} L_j^{(\alpha)}(x)$$

which immediately gives

$$D^q L_n^{(\alpha)}(x) = (-1)^q \sum_{j=0}^{n-q} \binom{n-j-1}{q-1} L_j^{(\alpha)}(x)$$

and this completes the proof of theorem 1. □

4. Laguerre coefficients of the moments of one single Laguerre polynomial of any degree

For the evaluation of Laguerre coefficients of the moments of higher-order derivatives of infinitely differentiable functions, the following theorem is needed.

Theorem 2.

$$x^m L_j^{(\alpha)}(x) = \sum_{n=0}^{2m} a_{mn}(j) L_{j+m-n}^{(\alpha)}(x) \qquad m \geqslant 0 \quad j \geqslant 0 \tag{9}$$

where

$$a_{mn}(j) = \frac{(-1)^{m-n}(m!)^2}{\Gamma(j+m-n+\alpha+1)} \sum_{k=\max(0,j-n)}^{\min(j+m-n,j)} \binom{j+m-n}{k} \frac{\Gamma(m+k+\alpha+1)}{(j-k)!(n-j+k)!(m-j+k)!}. \tag{10}$$

Proof. We use the induction principle to prove this theorem. In view of the recurrence relation (3), we may write

$$xL_j^{(\alpha)}(x) = -(j+1)L_{j+1}^{(\alpha)}(x) + (2j+\alpha+1)L_j^{(\alpha)}(x) - (j+\alpha)L_{j-1}^{(\alpha)}(x) \qquad j \geqslant 0$$

which may be put in the form

$$xL_j^{(\alpha)}(x) = a_{10}(j)L_{j+1}^{(\alpha)}(x) + a_{11}(j)L_j^{(\alpha)}(x) + a_{12}(j)L_{j-1}^{(\alpha)}(x) \tag{11}$$

this in turn shows that (9) is true for $m = 1$. Proceeding by induction, assuming that (9) is valid for m, we want to prove that

$$x^{m+1} L_j^{(\alpha)}(x) = \sum_{n=0}^{2m+2} a_{m+1,n}(j) L_{j+m-n+1}^{(\alpha)}(x). \tag{12}$$

From (11) and assuming the validity of (9) for m, we have

$$\begin{aligned} x^{m+1} L_j^{(\alpha)}(x) = \sum_{n=0}^{2m} a_{mn}(j)\big[&a_{10}(j+m-n)L_{j+m-n+1}^{(\alpha)}(x) \\ &+ a_{11}(j+m-n)L_{j+m-n}^{(\alpha)}(x) + a_{12}(j+m-n)L_{j+m-n-1}^{(\alpha)}(x)\big]. \end{aligned}$$

Collecting similar terms, we get

$$\begin{aligned} x^{m+1} L_j^{(\alpha)}(x) = {} & a_{m0}(j)a_{10}(j+m)L_{j+m+1}^{(\alpha)}(x) + [a_{m1}(j)a_{10}(j+m-1) \\ & + a_{m0}(j)a_{11}(j+m)]L_{j+m}^{(\alpha)}(x) + \sum_{n=2}^{2m}[a_{mn}(j)a_{10}(j+m-n) \\ & + a_{m,n-1}(j)a_{11}(j+m-n+1) + a_{m,n-2}(j)a_{12}(j+m-n+2)]L_{j+m-n+1}^{(\alpha)}(x) \\ & + [a_{m,2m}(j)a_{11}(j-m) + a_{m,2m-1}(j)a_{12}(j-m+1)]L_{j-m}^{(\alpha)}(x) \\ & + a_{m,2m}(j)a_{12}(j-m)L_{j-m-1}^{(\alpha)}(x). \end{aligned} \tag{13}$$

It can be easily shown that

$$\begin{aligned} a_{m+1,0}(j) &= a_{m0}(j)a_{10}(j+m) \\ a_{m+1,1}(j) &= a_{m1}(j)a_{10}(j+m-1) + a_{m0}(j)a_{11}(j+m) \\ a_{m+1,n}(j) &= a_{mn}(j)a_{10}(j+m-n) + a_{m,n-1}(j)a_{11}(j+m-n+1) \\ &\quad + a_{m,n-2}(j)a_{12}(j+m-n+2) \\ a_{m+1,2m+1}(j) &= a_{m,2m}(j)a_{11}(j-m) + a_{m,2m-1}(j)a_{12}(j-m+1) \\ a_{m+1,2m+2}(j) &= a_{m,2m}(j)a_{12}(j-m) \end{aligned}$$

and accordingly, formula (13) becomes

$$x^{m+1}L_j^{(\alpha)}(x) = \sum_{n=0}^{2m+2} a_{m+1,n}(j)L_{j+m-n+1}^{(\alpha)}(x)$$

which completes the induction and proves the theorem. □

It is worth noting that, recalling the definition of Pochhammer's symbol,

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = \frac{(-1)^n\Gamma(1-z)}{\Gamma(1-z-n)}$$

and the identity

$$\binom{n}{k} = \frac{(-1)^k(-n)_k}{k!}$$

formula (10) can be written in terms of a ${}_3F_2$ hypergeometric function of unit argument

$$a_{mn}(j) = \frac{(-1)^{m-n}(m!)^2\Gamma(m+\alpha+1)}{j!(n-j)!(m-j)!\Gamma(j+m-n+\alpha+1)}$$
$$\times\, {}_3F_2\left(\begin{matrix} -(j+m-n), & -j, & m+\alpha+1; \\ & & & ;1 \\ n-j+1, & m-j+1; & \end{matrix}\right).$$

Corollary 1. *It is not difficult to show that*

$$x^m L_j^{(\alpha)}(x) = \sum_{n=0}^{j+m} a_{m,j+m-n}(j)L_n^{(\alpha)}(x) \qquad j \geqslant 0 \quad m \geqslant 0 \tag{14}$$

and

$$x^m = \sum_{n=0}^{m} a_{m,m-n}(0)L_n^{(\alpha)}(x) \qquad m \geqslant 0 \tag{15}$$

where

$$a_{m,m-n}(0) = \frac{(-1)^n m!\Gamma(m+\alpha+1)}{(m-n)!\Gamma(n+\alpha+1)}.$$

This result is in complete agreement with that given in Rainville (1960, p 207).

5. Laguerre coefficients of a general-order derivative of an infinitely differentiable function

Theorem 3. *Let $f(x)$ and all its derivatives be smooth and $f(x)$ and $f^{(q)}(x)$ be expanded as in (5) and (6) respectively, and for a positive integer ℓ, let*

$$x^\ell \frac{\mathrm{d}^q f(x)}{\mathrm{d}x^q} = I^{q,\ell} \tag{16}$$

and if we write

$$I^{q,\ell} = \sum_{i=0}^{\infty} b_i^{q,\ell} L_i^{(\alpha)}(x) \tag{17}$$

then

$$b_i^{q,\ell} = \begin{cases} \sum_{k=0}^{\ell-1} a_{\ell,k+\ell-i}(k) a_k^{(q)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell) a_{k+\ell}^{(q)} & 0 \leqslant i \leqslant \ell \\ \sum_{k=i-\ell}^{\ell-1} a_{\ell,k+\ell-i}(k) a_k^{(q)} + \sum_{k=0}^{i} a_{\ell,k+2\ell-i}(k+\ell) a_{k+\ell}^{(q)} & \ell+1 \leqslant i \leqslant 2\ell-1 \\ \sum_{k=i-2\ell}^{i} a_{\ell,k+2\ell-i}(k+\ell) a_{k+\ell}^{(q)} & i \geqslant 2\ell. \end{cases} \tag{18}$$

Proof. Equations (6), (9) and (16) give

$$I^{q,\ell} = \sum_{k=0}^{\infty} a_k^{(q)} \sum_{j=0}^{2\ell} a_{\ell,j}(k) L_{k+\ell-j}^{(\alpha)}(x). \tag{19}$$

By letting $i = k + \ell - j$, (19) may be written in the form

$$\begin{aligned} I^{q,\ell} &= \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x) + \sum_{k=\ell}^{\infty} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x) \\ &= \sum_1 + \sum_2 \end{aligned} \tag{20}$$

where

$$\sum_1 = \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x)$$

$$\sum_2 = \sum_{k=\ell}^{\infty} a_k^{(q)} \sum_{i=k-\ell}^{k+\ell} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x).$$

Considering $\sum_1$ first,

$$\begin{aligned} \sum_1 &= \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{-1} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x) + \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=0}^{k+\ell} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x) \\ &= \sum_{11} + \sum_{12}. \end{aligned} \tag{21}$$

Clearly,

$$\sum_{11} = \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=k-\ell}^{-1} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x) = \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=1}^{\ell-k} a_{\ell,k+\ell+i}(k) L_{-i}^{(\alpha)}(x)$$

hence

$$\sum_{11} = 0. \tag{22}$$

Now,

$$\begin{aligned} \sum_{12} &= \sum_{k=0}^{\ell-1} a_k^{(q)} \sum_{i=0}^{k+\ell} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x) \\ &= \sum_{i=0}^{\ell} \sum_{k=0}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x) + \sum_{i=\ell+1}^{2\ell-1} \sum_{k=i-\ell}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x) \end{aligned}$$

hence,

$$\sum_{12} = \sum_{i=0}^{2\ell-1} \sum_{k=\max(0,i-\ell)}^{\ell-1} a_k^{(q)} a_{\ell,k+\ell-i}(k) L_i^{(\alpha)}(x). \tag{23}$$

Substitution of (22) and (23) into (21) yields

$$\sum_{1} = \sum_{i=0}^{2\ell-1} \sum_{k=\max(0,i-\ell)}^{\ell-1} a_k^{(q)} a_{k+\ell-i}(k) L_i^{(\alpha)}(x). \tag{24}$$

When considering $\sum_2$, if one takes $k+\ell$ instead of k, it is not difficult to show that

$$\sum_{2} = \sum_{i=0}^{\infty} \sum_{k=\max(0,i-2\ell)}^{i} a_{k+\ell}^{(q)} a_{\ell,k+2\ell-i}(k+\ell) L_i^{(\alpha)}(x). \tag{25}$$

Substitution of (24) and (25) into (20) gives the required results (18) and completes the proof of theorem 3. □

6. Application to ordinary differential equations with varying coefficients

Let $f(x)$ be an infinitely differentiable function defined on $[0,\infty)$ and having the Laguerre expansion (5), and assume that it satisfies the linear nonhomogeneous differential equation of order n,

$$\sum_{i=0}^{n} p_i(x) f^{(i)}(x) = p(x) \tag{26}$$

where $p_0, p_1, \ldots, p_n \neq 0$ are polynomials of x, and the coefficients of the Laguerre series of the function $p(x)$ are known; formulae (7), (9) and (18) enable one to construct, in view of equation (26), the linear recurrence relation of order r, namely

$$\sum_{j=0}^{r} \alpha_j(k) a_{k+j} = \beta(k) \qquad k \geqslant 0 \tag{27}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_r$ ($\alpha_0 \neq 0, \alpha_r \neq 0$) are polynomials of the variable k. The interested reader is referred to Doha (1998) for a similar derivation of (27) when the basis of expansion is ultraspherical polynomials.

An example dealing with a nonhomogeneous differential equation is considered in order to clarify application of the results obtained.

Example. Consider the nonhomogeneous differential equation

$$2xy'' + (1+4x)y' + (1+2x)y = \mathrm{e}^{-x} \qquad y(0) = 0 \qquad y'(0) = 1. \tag{28}$$

If e^{-x} is expanded in the form

$$\mathrm{e}^{-x} = \sum_{i=0}^{\infty} f_i L_i^{(\alpha)}(x) \tag{29}$$

then

$$f_i = \frac{1}{2^{i+\alpha+1}} \tag{30}$$

and if $y(x)$ is expanded in the form

$$y(x) = \sum_{i=0}^{\infty} a_i L_i^{(\alpha)}(x)$$

then by virtue of formulae (16) and (17), equation (28) takes the form

$$2b_i^{2,1} + b_i^{1,0} + 4b_i^{1,1} + b_i^{0,0} + 2b_i^{0,1} = f_i \qquad i \geqslant 0. \tag{31}$$

Formula (18) gives

$$\left.\begin{array}{ll} b_i^{q,1} = \sum_{k=i-2}^{i} a_{1,k+2-i}(k+1)a_{k+1}^{(q)} & q = 0, 1, 2 \\ b_i^{q,0} = a_i^{(q)} & q = 0, 1 \end{array}\right\} \qquad i \geqslant 0. \tag{32}$$

Substitution of relations (32) into equation (31), and making use of formulae (7) and (10)—after some manipulation—yields the following recurrence relation:

$$\begin{aligned} 2i[(2i+3)^2 + 2\alpha]a_{i-1} &- [16i^3 + 4i^2(9-2\alpha) - 16i\alpha - 4\alpha(\alpha+4) - 23]a_i \\ &- [40i^3 + 4i^2(10\alpha+43) + 2i(62\alpha+111) + (2\alpha+5)(10\alpha+13)]a_{i+1} \\ &+ 2[48i^3 + 4i^2(8\alpha+45) + 88i(\alpha+2) + 2\alpha(8\alpha+29) + 25]a_{i+2} \\ &- 8(i+\alpha+3)[(2i+1)^2 + 2\alpha]a_{i+3} \\ = {} &[(2i+1)^2 + 2\alpha]f_{i+2} - 2[4i(i+2) + 2\alpha + 1]f_{i+1} \\ &+ [(2i+3)^2 + 2\alpha]f_i \qquad i \geqslant 0. \end{aligned} \tag{33}$$

The complete solution of this example may be obtained by solving the recurrence relation (33). What is worth noting is that the analytical solution for this recurrence relation is given explicitly by

$$a_i = \frac{1+\alpha-i}{2^{\alpha+i+2}} \qquad i \geqslant 0. \tag{34}$$

An analytical solution such as (34) is not generally easy to obtain. The alternative approach for solving (33) can be obtained by using the well-known methods of Miller and Oliver as well as modifications and generalizations of these methods (see Jirari (1995), Luke (1969), Oliver (1988), Scraton (1972), Wimp (1984) and Weixlbaumer (2001)).

7. Recurrence relations for connection coefficients between Jacobi and Laguerre polynomials

Suppose V is a vector space of all polynomials over the real or complex numbers and V_m is the subspace of polynomials of degree less than or equal to m. Suppose $p_0(x), p_1(x), p_2(x), \ldots$ is a sequence of polynomials such that $p_n(x)$ is of exact degree n; let $q_0(x), q_1(x), q_2(x), \ldots$ be another such sequence. Clearly, these sequences form a basis for V. It is also evident that $p_0(x), p_1(x), \ldots, p_m(x)$ and $q_0(x), q_1(x), \ldots, q_m(x)$ give two bases for V_m. While working with finite-dimensional vector spaces, it is often necessary to find the matrix that transforms a basis of a given space to another basis. This means that one is interested in the connection coefficients $a_i(n)$ that satisfy

$$q_n(x) = \sum_{i=0}^{n} a_i(n)p_i(x). \tag{35}$$

The choice of $p_n(x)$ and $q_n(x)$ depends on the situation. For example, suppose

$$p_n(x) = x^n \qquad q_n(x) = x(x-1)\cdots(x-n+1)$$

then the connection coefficients $a_i(n)$ are Stirling numbers of the first kind. If the roles of these $p_n(x)$ and $q_n(x)$ are interchanged, we get Stirling numbers of the second kind. These numbers are useful in some combinatorial polynomials (see Abramowitz and Stegun (1970), pp 824–5).

Usually, little can be said about these connection coefficients. However, there are some cases where simple formulae can be obtained (see, for instance, Andrews *et al* (1999)). The aim of this section is to describe a simple procedure (based on the results of theorem 3) in order to find recurrence relations, sometimes easy to solve, between the coefficients $a_i(n)$ when $p_i(x) = L_i^{(\alpha)}(x)$ and $q_n(x) = P_n^{(\gamma,\delta)}(x)$, where $L_i^{(\alpha)}(x)$ and $P_n^{(\gamma,\delta)}(x)$ are the Laguerre and Jacobi orthogonal polynomials. This gives an alternative way to be compared to the approaches of Koepf and Schmersau (1998), Lewanowicz (2002), Lewanowicz and Woźny (2001), Lewanowicz *et al* (2000), Ronveaux *et al* (1995), Godoy *et al* (1997) and Sánchez-Ruiz and Dehesa (1998). A nonrecursive way to approach the problem in the case of classical orthogonal polynomials of discrete variable can be found in Gasper (1974). Moreover, other authors have considered the problem from a recursive point of view (see Koepf and Schmersau (1988)), or even in classical discrete and q-analogues (cf Álvarez-Nodarse *et al* (1997) and Álvarez-Nodarse and Ronveaux (1996)).

7.1. The Jacobi–Laguerre connection problem

The link between $P_n^{(\gamma,\delta)}(x)$ and $L_i^{(\alpha)}(x)$ given by (35) can easily be replaced by a linear relation involving only $L_i^{(\alpha)}(x)$ using the Jacobi differential equation, namely,

$$(1-x^2)D^2P_n^{(\gamma,\delta)}(x) + [\delta-\gamma-(2+\delta+\gamma)x]DP_n^{(\gamma,\delta)}(x) + n(1+\gamma+\delta+n)P_n^{(\gamma,\delta)}(x) = 0 \tag{36}$$

by substituting

$$P_n^{(\gamma,\delta)}(x) = \sum_{i=0}^{\infty} a_i(n)L_i^{(\alpha)}(x) \tag{37}$$

with $a_{n+1}(n) = a_{n+2}(n) = \cdots = 0$. By virtue of formulae (16) and (17), equation (36) takes the form

$$I^{2,0} - I^{2,2} - (2+\gamma+\delta)I^{1,1} + (\delta-\gamma)I^{1,0} + n(1+\gamma+\delta+n)I^{0,0} = 0$$

or

$$b_i^{2,0} - b_i^{2,2} - (2+\gamma+\delta)b_i^{1,1} + (\delta-\gamma)b_i^{1,0} + n(1+\gamma+\delta+n)b_i^{0,0} = 0. \tag{38}$$

Formula (18) gives

$$b_i^{2,0} = a_i^{(2)}(n) \qquad b_i^{1,0} = a_i^{(1)}(n) \qquad b_i^{0,0} = a_i(n) \qquad i \geqslant 0 \tag{39}$$

$$b_i^{1,1} = \sum_{k=i-2}^{i} a_{1,k+2-i}(k+1)a_{k+1}^{(1)} \qquad b_i^{2,2} = \sum_{k=i-4}^{i} a_{2,k+4-i}(k+2)a_{k+2}^{(2)} \qquad i \geqslant 0. \tag{40}$$

Substitution of relations (39) and (40) into equation (38) and making use of formulae (7) and (10)—after some little manipulation—yields the following recurrence relation:

$$\begin{aligned}(n-i)(n+i+\gamma+\delta+1)a_i(n) = {} & [2(n-i-1)(n+i+\gamma+\delta+2) - (i+\alpha+1)\\ & \times(2i+\gamma+\delta+2) + \delta-\gamma]a_{i+1}(n) + [(i+\alpha+2)(5i+2\gamma+2\delta+\alpha+9)\\ & -(n-i-2)(n+i+\gamma+\delta+3) + \gamma-\delta-1]a_{i+2}(n) - (i+\alpha+3)\\ & \times(4i+\gamma+\delta+2\alpha+10)a_{i+3}(n) + (i+\alpha+3)(i+\alpha+4)a_{i+4}(n)\\ & i = n-1, n-2, \ldots, 0\end{aligned} \tag{41}$$

which is of order 4. It is to be noted here that the fourth-order recurrence relation (41) generates the coefficients $a_i(n)$ of (30) by recurring backward with the initial conditions given by

$$a_{n+s}(n) = 0 \qquad s = 1, 2, 3 \text{ and } a_n(n) = \frac{(-1)^n\Gamma(2n+\gamma+\delta+1)}{2^n\Gamma(n+\gamma+\delta+1)}.$$

The coefficient $a_n(n)$, which only depends on the relative normalization of $P_n^{(\gamma,\delta)}(x)$ and $L_n^{(\alpha)}(x)$, has been easily obtained by identification of the highest power in expansion (37).

The solution of (41) is

$$a_i(n) = \sum_{j=i}^{n} A_j(n) B_{ji} \tag{42}$$

where

$$A_j(n) = \frac{2^j (1+\alpha)_j (1+\gamma)_n}{(1+\gamma+\delta)_n} \sum_{\substack{k=j \\ (k-j)\ \text{even}}}^{n} \frac{(-1)^{\frac{2n-j-k}{2}} (\delta-\gamma)_{n-k} (1+\gamma+\delta)_{n+k} (1+2\gamma+2k)}{k!(n-k)!(1+2\gamma)_{n+k+1} \left(\frac{k-j}{2}\right)!}$$
$$\times \left(\gamma+\frac{1}{2}\right)_{\frac{j+k}{2}} {}_2F_0\left(-\left(\frac{k-j}{2}\right), \gamma+\frac{k+j+1}{2}, -, 1\right) \qquad (n-j)\ \text{even} \tag{43}$$

$$B_{ji} = \frac{(-j)_i}{(1+\alpha)_i} {}_2F_2\begin{pmatrix} -\frac{1}{2}(j-i), & -\frac{1}{2}(j-i-1); & \\ & & ;-\frac{1}{4} \\ -\frac{1}{2}(\alpha+j), & -\frac{1}{2}(\alpha+j-1); & \end{pmatrix}. \tag{44}$$

Now, we consider the ultraspherical-Laguerre connection problem and its consequences, Chebyshev of the first kind-Laguerre, Chebyshev of the second kind-Laguerre and Legendre-Laguerre. These are given in the following corollaries:

Corollary 2. *In the connection problem*

$$C_n^{(\lambda)}(x) = \frac{n!\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma\left(n+\lambda+\frac{1}{2}\right)} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x) = \sum_{i=0}^{n} a_i(n) L_i^{(\alpha)}(x)$$

with $\gamma = \delta = \lambda - \frac{1}{2}, -\frac{1}{2} < \lambda < \infty, C_n^{(\lambda)}(1) = 1$, *where* $C_n^{(\lambda)}(x)$ *are the ultraspherical polynomials, the* $a_i(n)$*-coefficients are given by*

$$a_i(n) = \sum_{j=i}^{n} A_j(n) B_{ji} \tag{45}$$

where

$$A_j(n) = \frac{2^j (1+\alpha)_j \lambda_{\frac{j+n}{2}}}{\left(\frac{n-j}{2}\right)!(2\lambda)_n} {}_2F_0\left(-\left(\frac{n-j}{2}\right), \lambda+\frac{n+j}{2}, -, 1\right) \qquad (n-j)\ \textit{even}.$$

Corollary 3. *In the connection problem*

$$T_n(x) = C_n^{(0)}(x) = \sum_{i=0}^{n} a_i(n) L_i^{(\alpha)}(x)$$

where $T_n(x)$ *are Chebyshev polynomials of the first kind, the coefficients* $a_i(n)$ *are given by*

$$a_i(n) = \sum_{j=i}^{n} A_j(n) B_{ji} \tag{46}$$

where

$$A_j(n) = \frac{2^j (1+\alpha)_j}{\left(\frac{n-j}{2}\right)!} {}_2F_0\left(-\left(\frac{n-j}{2}\right), \frac{n+j}{2}, -, 1\right) \qquad (n-j)\ \textit{even}.$$

Corollary 4. *In the connection problem*

$$P_n(x) = C_n^{(\frac{1}{2})}(x) = \sum_{i=0}^{n} a_i(n) L_i^{(\alpha)}(x)$$

where $P_n(x)$ are Legendre polynomials, the coefficients $a_i(n)$ are given by

$$a_i(n) = \sum_{j=i}^{n} A_j(n) B_{ji} \tag{47}$$

where

$$A_j(n) = \frac{2^j (1+\alpha)_j \left(\frac{1}{2}\right)_{\frac{j+n}{2}}}{\left(\frac{n-j}{2}\right)! n!} {}_2F_0\left(-\left(\frac{n-j}{2}\right), \frac{n+j+1}{2}, -, 1\right) \qquad (n-j) \text{ even.}$$

Corollary 5. *In the connection problem*

$$U_n(x) = (n+1) C_n^{(1)}(x) = (n+1) \sum_{i=0}^{n} a_i(n) L_i^{(\alpha)}(x)$$

where $U_n(x)$ are Chebyshev polynomials of the second kind, the coefficients $a_i(n)$ are given by

$$a_i(n) = \sum_{j=i}^{n} A_j(n) B_{ji} \tag{48}$$

where

$$A_j(n) = \frac{2^j (1+\alpha)_j \left(\frac{n+j}{2}\right)!}{\left(\frac{n-j}{2}\right)! n!} {}_2F_0\left(-\left(\frac{n-j}{2}\right), \frac{n+j+2}{2}, -, 1\right) \qquad (n-j) \text{ even.}$$

Note. *The coefficients B_{ji} in relations (45)–(48) are defined by (44).*

7.2. The Hermite–Laguerre connection problem

In this problem

$$H_n(x) = \sum_{i=0}^{n} a_i(n) L_i^{(\alpha)}(x) \tag{49}$$

where $H_n(x)$ are Hermite polynomials, which satisfy the differential equation

$$D^2 H_n(x) - 2x D H_n(x) + 2n H_n(x) = 0.$$

The coefficients $a_i(n)$ satisfy the third-order recurrence relation

$$\begin{aligned} 2(n-i)a_i(n) = {} & 2(2n-3i-\alpha-3)a_{i+1}(n) - (2n-6i-4\alpha-11)a_{i+2}(n) \\ & - 2(i+\alpha+3)a_{i+3}(n) \qquad i = n-1, n-2, \ldots, 0 \end{aligned} \tag{50}$$

with $a_{n+s}(n) = 0$, $s = 1, 2$ and $a_n(n) = (-1)^n 2^n n!$. The solution of (50) is

$$a_i(n) = \frac{(-n)_i 2^n (1+\alpha)_n}{(1+\alpha)_i} {}_2F_2\left(\begin{matrix} -\frac{1}{2}(n-i), & -\frac{1}{2}(n-i-1); \\ & & ; -\frac{1}{4} \\ -\frac{1}{2}(\alpha+n), & -\frac{1}{2}(\alpha+n-1); \end{matrix}\right).$$

Remark 1. It is worth mentioning that the recurrence relations (41) and (49) are minimal (i.e. the shortest ones in order) for the connection coefficients in (37) and (49) respectively. These are in agreement with the results of Godoy *et al* (1997), displayed in table 1, p 263.

Remark 2. It should be mentioned that our goal here is to emphasize the systematic character and simplicity of our algorithm, which allows one to implement it in any computer algebra (here the Mathematica (1999) symbolic language has been used).

To end this paper, we wish to report that this work deals with formulae associated with the Laguerre coefficients for the moments of a general-order derivative of differentiable functions and with the connection coefficients between Jacobi–Laguerre and Hermite–Laguerre polynomials. These formulae can be used to facilitate greatly the setting up of the algebraic systems to be obtained by applying the spectral or pseudospectral methods for solving differential equations with polynomial coefficients of any order.

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References

Abramowitz M and Stegun I A (ed) 1970 *Handbook of Mathematical Functions (Appl. Maths. Series vol 55)* (New York: National Bureau of Standards)

Álvarez-Nodarse R and Ronveaux A 1996 Recurrence relations for connection coefficients between q-orthogonal polynomials of discrete variables in the non-uniform lattice $x(s) = q^{2s}$ *J. Phys. A: Math. Gen.* **29** 7165–75

Álvarez-Nodarse R, Yáñez R J and Dehesa J S 1997 Modified Clebsch–Gordan-type expansions for products of discrete hypergeometric polynomials *J. Comput. Appl. Math.* **89** 171–97

Andrews G E, Askey R and Roy R 1999 *Special Functions* (Cambridge: Cambridge University Press)

Bagrov V G and Gitman D M 1990 *Exact Solutions of Relativistic Wave Equations* (Dordrecht: Kluwer)

Canuto C, Hussaini M Y, Quarteroni A and Zang T A 1988 *Spectral Methods in Fluid Dynamics* (Berlin: Springer)

Doha E H 1991 The coefficients of differentiated expansions and derivatives of ultraspherical polynomials *J. Comput. Math. Appl.* **21** 115–22

Doha E H 1994 The first and second kinds Chebyshev coefficients of the moments of the general-order derivative of an infinitely differentiable function *Int. J. Comput. Math.* **51** 21–35

Doha E H 1998 The ultraspherical coefficients of the moments of a general-order derivative of an infinitely differentiable function *J. Comput. Appl. Math.* **89** 53–72

Doha E H 2002 On the coefficients of differentiated expansions and derivatives of Jacobi polynomials *J. Phys. A: Math. Gen.* **35** 3467–78

Doha E H 2003 On the connection coefficients and recurrence relations arising from expansions in series of Hermite polynomials *Integral Transforms and Special Functions* (accepted)

Doha E H and Abd-Elhameed W M 2002 Efficient spectral Galerkin-algorithms for direct solution of second-order equations using ultraspherical polynomials *SIAM J. Sci. Comput.* **24** 548-71

Doha E H and El-Soubhy S I 1995 On the Legendre coefficients of the moments of the general-order derivative of an infinitely differentiable function *Int. J. Comput. Math.* **56** 107–22

Fox L and Parker I B 1972 *Chebyshev Polynomials in Numerical Analysis* (London: Oxford University Press)

Gasper G 1974 Projection formulas for orthogonal polynomials of a discrete variable *J. Math. Anal.* **45** 176–98

Godoy E, Ronveaux A, Zarzo A and Area I 1997 Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: continuous case *J. Comput. Appl. Math.* **84** 257–75

Gottlieb D and Orszag S A 1977 *Numerical Analysis of Spectral Methods: Theory and Applications (CBMS—NSF Regional Conf. Series in Applied Mathematics vol 26)* (Philadelphia, PA: Society of Industrial and Applied Mathematics) pp 74–100

Jirari A 1995 Second-order Sturm–Liouville difference equations and orthogonal polynomials *Mem. Am. Math. Soc.* **113**

Karageorghis A 1988a A note on the Chebyshev coefficients of the general-order derivative of an infinitely differentiable function *J. Comput. Appl. Math.* **21** 129–32

Karageorghis A 1988b A note on the Chebyshev coefficients of the moments of the general-order derivative of infinitely differentiable function *J. Comput. Appl. Math.* **21** 383–6

Koepf W and Schmersau D 1998 Representations of orthogonal polynomials *J. Comput. Appl. Math.* **90** 57–94

Lanczos C 1957 *Applied Analysis* (London: Pitman)

Lewanowicz S 2002 Recurrences for the coefficients of series expansions with respect to classical orthogonal polynomials *Appl. Math.* **29** 97–116

Lewanowicz S, Godoy E, Area I, Ronveaux A and Zarzo A 2000 Recurrence relations for the coefficients of the Fourier series expansions with respect to q-classical orthogonal polynomials *Numer. Algorithms* **23** 31–50

Lewanowicz S and Woźny P 2001 Algorithms for construction of recurrence relations for the coefficients of expansions in series of classical orthogonal polynomials *Preprint* Inst. of Computer Sci., Univ. of Wroclaw Feb. 2001 webpage http://www.ii.uni.wroc.pl/sle/publ.html

Luke Y 1969 *The Special Functions and Their Approximations* vol 1 (New York: Academic)

Nikiforov A F and Uvarov V B 1988 *Special Functions in Mathematical Physics* (Basel: Birkhauser)

Oliver J 1969 The numerical solution of linear recurrence relations *Numer. Math.* **11** 349–60

Phillips T N 1988 On the Legendre coefficients of a general-order derivative of an infinitely differentiable function *IMA J. Numer. Anal.* **8** 455–9

Rainville E D 1960 *Special Functions* (New York: Macmillan)

Ronveaux A, Zarzo A and Godoy E 1995 Recurrence relations for connection coefficients between two families of orthogonal polynomials *J. Comput. Appl. Math.* **62** 67–73

Sánchez-Ruiz J and Dehesa J S 1998 Expansions in series of orthogonal polynomials *J. Comput. Appl. Math.* **89** 155–70

Scraton R E 1972 A modification of Miller's recurrence algorithms *BIT* **12** 242–51

Weixlbaumer C 2001 Solutions of difference equations with polynomial coefficients *RISC* Linz Johannes Kepler Universität, Australia

Wimp J 1984 *Computation with Recurrence Relations* (London: Pitman)

Wolfram Research 1999 Inc. Mathematica version 4.0 (Wolfram Research, Champaign, IL)